

No-Signaling in Steepest Entropy Ascent: A Nonlinear Non-local Non-equilibrium Quantum Dynamics of Composite Systems

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The Lindbladian formalism models open quantum systems using a ‘bottom-up’ approach, deriving linear dynamics from system-environment interactions. We present a ‘top-down’ approach starting with phenomenological constraints, focusing on system’s structure, subsystems’ interactions, environmental effects, and often using a non-equilibrium variational principle designed to enforce strict thermodynamic consistency. However, incorporating the second law’s requirement — that Gibbs states are the sole stable equilibria — necessitates nonlinear dynamics, challenging no-signaling principles in composite systems. We reintroduce ‘local perception operators’ and show that they allow to model signaling-free non-local effects. Using the steepest-entropy-ascent variational principle as an example, we demonstrate the validity of the ‘top-down’ approach for integrating quantum mechanics and thermodynamics in phenomenological models, with potential applications in quantum computing and resource theories.

1 Introduction

The study of open-quantum systems can be categorized into two general approaches. The prevailing of the two is based on the celebrated Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation under the assumption that the system is weakly coupled to the background environment (for recent reviews and generalizations, see,

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e.g., [1, 2]). Various thermodynamic concepts are used to theorize in this paradigm [3], including but not limited to finite-size heat baths [4] and reinterpretations of the second law of thermodynamics [5]. Here, we start from the Schrödinger equation of the system-background composite in a pure state and trace out the background (or a part of it) to model irreversible dynamics for the system under observation. As emphasized by Kosloff [1], a suitable set of assumptions is required, but the key idea is that initially, the system and background start in an uncorrelated product state. Then further assumptions are added to arrive at a satisfactory model master equation which is ‘essentially compatible with the laws of thermodynamics.’ We dub this traditional approach ‘bottom-up’. It starts from the fundamental dogma that the universe is a closed system evolving unitarily and the heuristic dogma that any quantum system is influenced by its environment. Then, adding enough ‘reasonable’ (modeling) assumptions, we finally derive a thermodynamically consistent effectively phenomenological master equation for the density operator of the local system (or system plus the immediate portion of the environment strongly coupled to it).

For four decades, an alternative phenomenological approach has existed but remains seldom used. For contrast, we call this a ‘top-down’ approach. It assumes that a physical phenomenon is described by local model dynamics for the density operator of the system, which may comprise multiple subsystems and include strongly coupled parts of the environment. We strive to find the most general master equation that accounts for the composite system’s features and subsystem interactions while ensuring strong compatibility with thermodynamics. Importantly, if we seek

to incorporate the second-law requirement that Gibbs states be the only conditionally stable equilibrium states of the dynamics [6, 7] — a thermodynamic compatibility requirement seldom considered and often overlooked in the bottom-up approach — then the condition that the final model dynamics be linear in density operator must be removed. In other words, the top-down approach opens up to nonlinear models. In their work, Simmons and Park [6] show that for such an approach to incorporate a successful merger of thermodynamics and mechanics, non-linearity in the dynamics is an essential requirement ¹.

Consider the recent experimental work by Somhorst *et al.* [10], where the commonly accepted view of emergent thermodynamics from the openness of a local system has been demonstratively validated. This evidence reinforces the need for phenomenological top-down models equipped with strong thermodynamic consistency. We claim this because, most of the time, experimenters do not have such full control of the environment of the local system of interest as in the cited experiment, and most of the time, very little is known about the details of the actual environment and its coupling with the system. Yet, to model the local thermodynamic evolution employing linear Kraus maps, it is necessary to concoct the proper Lindblad operators, which in turn require a model of the immediate environment. Instead, top-down models can incorporate the known physics of the system and model phenomenologically the ‘apparent decoherence due to distribution of information among the environmental degrees of freedom.’ They can do so by a minimal set of empirical parameters, and can accommodate situations where more physical (possibly non-commuting [11]) observables are conserved at the local level than just the mean energy (like in some experiments in [10]).

A powerful family of top-down master equations originates from a nonlinear extension of

¹In a related context, we note that recent experiments testing the nonlinear quantum mechanics (NQM) [8, 9] have placed constraints on potential deviations from standard quantum mechanics, but the results remain typically at a 90% confidence level. While we do not explore NQM in this work, it is important to clarify that our formalism is not part of this class. Instead, it is a phenomenological approach aimed at integrating quantum mechanics and thermodynamics to describe irreversible dynamics and ensure thermodynamic consistency at a fundamental level.

quantum mechanics initiated by Hatsopoulos and Gyftopoulos [12–15], and developed by Beretta [16–25]. This formalism originally sought to establish the second law of thermodynamics as a foundational principle at par with other conservation laws, embedded in a nonlinear law of evolution for mixed density operators, thus conceptualizing stability of canonical Gibbs states and spontaneous decoherence. Later, the steepest entropy ascent (SEA) formalism was developed and used successfully as a phenomenological modeling tool, applied to various quantum systems [26–28] including quantum computing [29, 30]. It was even elevated to the conceptual status of the fourth law of thermodynamics [31, 32] because of its equivalence with most of the more recent far-non-equilibrium thermodynamic modeling non-quantal formalisms.

In this paper, we aim to explicitly show that despite the embedded non-linearity in the equation of motion of SEA model dynamics, the evolution of a composite system is free from exotic interactions responsible for signaling between the non-interacting subsystems. Although the authors in Refs. [22, 24, 25, 27] claim that the SEA formalism abides by the no-signaling criteria (more on this later), it demands a discussion on its own as the claim is not a trivial one.

To attract future discussion on this topic, that may bear relevance also for current non-equilibrium modeling efforts in the quantum computing framework, we have structured the paper in the following form. In the next Section 2, we delve deeper into the meaning of signaling as established in the literature [33–40] and how it relates to nonlinearity. Thereafter, in Section 3, we explore the philosophical background that gave birth to the original SEA nonlinear law of evolution. In Section 4, we show how a nonlinear dynamics would help alleviate a still unresolved conundrum about the concept of individual quantum state. In Section 5, we discuss the measure-theoretic representation of mixed ensembles that allows to treat unambiguously nonlinearities and mixing of ensembles. We follow this by reintroducing and discussing in Section 6 the concept of local perception operators (LPO), and showing in Section 7 their crucial role in the construction of nonlinear and non-local model dynamics not affected by signaling issues. In Section 8 we discuss the system’s structure dependence of

the composite-system version of the SEA equation of motion and we argue that, by construction, its nonlinearities and non-localities do not cause signaling issues. In Section 9 we show how the general SEA equation of motion can be derived from a special implementation of the maximum entropy production variational principle. In Section 10 we derive the simplest version of the composite-system SEA equation of motion. In Section 11, we finally consider some explicit examples to validate our argument and illustrate pictorially some features of SEA evolution. In Section 12, we summarize our conclusions.

2 No-signaling and nonlinearity

Quantum mechanics (QM), as described by the Schrödinger-von Neumann formalism, is traditionally characterized by its linearity in state space and time evolution. Mean values of global and local properties and their time evolutions under unitary (and completely positive) dynamics, are linear functionals of the density operator. This linearity implies no-cloning, first shown by Park [41] and later rediscovered by Ghirardi [42], Wootters and Zurek [43], and Dieks [44]. Linearity also implies no-signaling [45–48]. However, Weinberg questioned this linear characteristic of QM [49]. He introduced nonlinearity in the operator formalism of the Schrödinger equation while preserving order-one homogeneity. This was one of the earliest examples of what is now broadly referred to as nonlinear quantum mechanics (NQM). Gisin [33] showed that such formalism could facilitate faster-than-light communication, *viz.*, signaling. Polchinski [34] proposed an alternative method to prevent signaling by restricting local evolutions to depend solely on the corresponding reduced density operators. However, this formalism still allowed signaling between different branches of the wavefunction (Everett telephone) [34]. Wódkiewicz and Scully [35] analyzed the nonlinear evolution of two-level atoms, with solutions later shown to depend on interpretation [36]. Czachor also showed that a nonlinear operator induces mobility phenomena (non-conservation of the inner-product of two pure states)[50].

Nonlinearity introduced via stochastic QM through Lindblad operator formalism [51] for

open quantum systems has been shown to respect no-signaling. Weinberg-inspired nonlinearities, interestingly, enable faster algorithms to solve *NP*-complete problems in polynomial time [37]. Later work [38] demonstrated that nonlinearity in QM can be incorporated without violating no-signaling, as long as time evolution is nonlinear while state space and operators remain linear. More recently, convex quasilinear maps [39, 52] were shown to support nonlinear QM dynamics without signaling, preserving key features of QM. Rembieliński and Caban [39] identified this as the minimal allowable deviation from QM’s linear structure. In a recent work, Kaplan and Rajendran [53] showed that in a low-temperature limit, the non-linearity introduced in quantum field theory results in causal nonlinear quantum mechanics, which is also another version of NQM.

Therefore, introducing nonlinearity in quantum mechanics (QM) can lead to exotic interactions that are difficult to justify physically. Nonlinear theories, as inspired by the Weinberg approach, allow for entropy oscillation [50]. Stochastic, jump-induced mixing of pure states [51] offers a basis for developing a thermodynamically consistent nonlinear master equation for the density operator of an open system. However, the search for theoretically consistent nonlinear models of quantum thermodynamics remains an open challenge. This is where we claim — and later demonstrate in the following sections — that the ‘top-down’ approach of SEA resolves the signaling problem entirely (pictorially sketched in Figure 1). From a philosophical perspective, SEA also addresses other conceptual conundrums, in particular the Schrödinger-Park paradox [54, 55], as already discussed by Park [56] and Beretta [57].

The no-signaling condition, as noted in [38], is typically enforced by requiring that, in the absence of mutual interactions between subsystems A and B, the evolution of A’s local observables depends solely on its reduced state. Formally, we express this as:

$$\frac{d\rho_J}{dt} = f(\rho_J), \quad (1)$$

where ρ_J is the reduced density operator (local state) of subsystem J . The SEA formalism, however, adopts a less restrictive perspective [22]. It

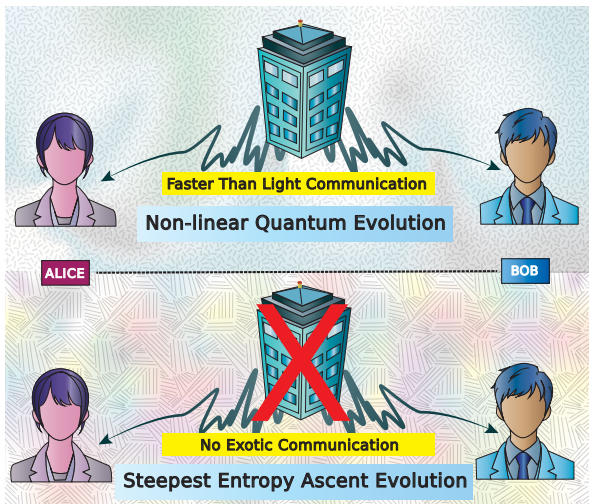


Figure 1: Schematic representation of no-signaling situation in the case of nonlinear quantum evolution. Alice and Bob are correlated but are non-interacting in a non-linear quantum theory (we imply the nonlinearities by the texture of the embedding background in the schematic). In the top part, a non-linear quantum evolution as described by Polchinski [34] or Gisin [33] may entail faster-than-light communication in some form (denoted by the telephone booth, see text for details). In the bottom part, instead, the embedding nonlinear steepest-entropy-ascent evolution cannot establish a similar exotic communication.

requires that if A and B are non-interacting, the law of evolution must not permit a local operation within B to influence the time evolution of A’s local (reduced, marginal) state. Consequently, the condition $\rho_A = \rho'_A$, applied to the two different states $\rho \neq \rho_A \otimes \rho_B$ and $\rho' = \rho_A \otimes \rho_B$, does not necessitate $d\rho_A/dt = d\rho'_A/dt$. This is because the local evolution can still be influenced by past interactions, such as existing entanglement and correlations, without violating the no-signaling principle. This highlights two key ideas: (1) analyzing local evolutions allows detection of correlations, but only those that can be classically communicated between subsystems, and (2) in the absence of interactions, nonlinear dynamics can cause correlations to diminish (spontaneous decoherence) but cannot create new correlations. We formally express this no-signaling condition as follows (see Section 8 for details),

$$\frac{d\rho_J}{dt} = f(\rho_J, (C_k)^J), \quad (2)$$

where $(C_k)^J$ are ‘local perception operators’ (LPO) (see Section 6 for their definition).

3 Philosophical motivations leading to SEA

Philosophically, SEA evolution was designed to conceive the irreversible equilibration and decoherence as a fundamental spontaneous dynamical feature, contrary to the coarse-graining approach. Entropy, the second law, and irreversibility could attain a more fundamental status as deterministic outcomes of quantal evolution [58], without contradicting standard QM. From the outset [16, 17, 59], nonlinearity was recognized as essential [6, 60] for this purpose. While the prevalent notion was — and still is — that the second law is emerging and statistical, the pioneers of SEA believed that the many ‘knots of thermodynamics’ [60, 61] could, at least conceptually, be untied by elevating it to a more fundamental stature. This motivated Hatsopoulos and Gyftopoulos (HG) to develop a unified theory of mechanics and thermodynamics [12–15]. They laid the foundation for what is now known as resource theory in quantum thermodynamics by extending ontological quantum states to include both pure and mixed density operators. The HG unified (resource) theory explicitly defines and fully characterizes the concept — later rediscovered as ‘ergotropy’ [62] — of maximum work extractable in a unitary process with cyclic parameter changes (theorems 3.2 and 3.3 in [13], which pioneered also the use of ‘majorization’ in quantum theory).

Beretta’s doctoral thesis [63] aimed to complete the HG theory with two key goals: 1) to design an equation of motion that establishes the second law as a theorem, ensuring Gibbs states are the only stable equilibrium states; and 2) to develop an unambiguous statistical representation of heterogeneous ensembles formed by statistical mixing of homogeneous (pure) ensembles² as one-to-one with the full set of pure and mixed density

²In the context of quantum thermodynamics, an ‘unambiguous statistical representation of heterogeneous ensembles’ ensures a clear distinction between intrinsic quantum uncertainties and classical statistical mixing. This approach, as developed by Beretta [63], treats heterogeneous ensembles as distributions over the full set of pure and mixed density operators, avoiding ambiguities associated with von Neumann’s statistical interpretation and resolving the Schrödinger-Park paradox. More on this in Section 5.

operators. The outcome of the first objective was the SEA formalism, including the introduction of local perception operators. The second objective led to the representation of generic ensembles using measure-theoretic distributions (see Section 5 for details) over the set of ontological states. The SEA formalism proved to be both robust and versatile, making it a powerful tool for thermodynamically consistent non-equilibrium modeling. Its applications expanded beyond its original scope to include phenomenological models in areas such as decoherence [64], mesoscopic transport rheology [65, 66], phase transitions [67], and quantum computing [28–30, 68].

4 Nonlinear dynamics and the ontic state conundrum

From a foundational perspective, the HG approach of assigning ontological status to the full set of density operators, pure and mixed, offers a potential resolution to the Schrödinger-Park paradox on the ambiguity of individual quantum states [54, 55, 57]. As noted by Park [56], treating density operators as one-to-one with homogeneous ensembles resolves ambiguity under two conditions: 1) that they evolve unitarily for short times, with nonlinear dynamics dominating over longer periods; and 2) that a full tomography, even for heterogeneous preparations, must include both linear and nonlinear observables, such as entropy, adiabatic availability, ergotropy, and free energy.

The von Neumann prescription [69, Chapter III] assigns a density operator to a statistical mixture of pure states, with mixing coefficients reflecting ignorance. However, Schrödinger [54] questioned this from the outset due to the non-unique decomposition of a mixed density operator into weighted sums of pure states. von Neumann’s approach irrecoverably blends quantal probabilities with non-quantal statistical weights, creating the ambiguity highlighted by Schrödinger and Park. To understand how a law of nonlinear evolution may resolve this issue, let $\hat{\Pi}_1^0$ and $\hat{\Pi}_2^0$ be two distinct homogeneous preparations. At time $t = 0$, they generate two initial ensembles of identically prepared, strictly isolated, and non-interacting systems. The ‘hat’ on Π denotes homogeneity of the

preparation, and superscript ‘0’ refers to time $t = 0$. According to the traditional statistical formulation by von Neumann, a preparation $\hat{\Pi}$ is *homogeneous* if and only if it cannot be replicated by any statistical composition of two *different* preparations. The statistics of measurement outcomes from a homogeneous preparation are represented by a pure (idempotent) density operator, $\hat{\rho}$, satisfying $\hat{\rho} = \hat{\rho}^2$. Consider a third (heterogeneous) preparation Π_3^0 obtained by the statistical mixture of $\hat{\Pi}_1^0$ and $\hat{\Pi}_2^0$ with fixed weight $0 < w < 1$, so that, formally, we may write $\Pi_3^0 = w\hat{\Pi}_1^0 + (1-w)\hat{\Pi}_2^0$. To obtain a quantum statistical mechanics (QSM) consistent with Gibbs-Boltzmann, Fermi-Dirac, and Bose-Einstein equilibrium distributions, von Neumann postulated that a heterogeneous preparation, such as Π_3^0 , should be represented by a mixed density operator $\rho_3^0 = w\hat{\rho}_1^0 + (1-w)\hat{\rho}_2^0$. This formulation, partly due to the linearity of quantum mechanics, has served as the foundation for successful frameworks such as equilibrium QSM and the Lindbladian theory of open systems.

These successes often lead us to overlook the Schrödinger-Park paradox or accept it as a seemingly enigmatic aspect of quantum theory. It arises when interpreting a heterogeneous ensemble. If we assume each member of the ensemble is prepared in either state $\hat{\rho}_1^0$ (with probability w) or $\hat{\rho}_2^0$ (with probability $1-w$), excluding all other states, we face a contradiction. The same mixed density operator ρ_3^0 allows infinitely many alternative decompositions. For instance, the decomposition, $\rho_3^0 = w'\hat{\rho}_4^0 + (1-w')\hat{\rho}_5^0$ suggests that the ensemble consists only of $\hat{\rho}_4^0$ and $\hat{\rho}_5^0$, excluding $\hat{\rho}_1^0$ and $\hat{\rho}_2^0$. This inconsistency challenges the notion of individual states in a heterogeneous ensemble and complicates the interpretation of QSM. Under a linear evolution map for the homogeneous preparations, $\hat{\Pi}^t = \mathcal{L}_t(\hat{\Pi}^0)$, and mixing of homogeneous preparations with time-invariant weights, we have

$$\Pi_3^t = w\hat{\Pi}_1^t + (1-w)\hat{\Pi}_2^t = w\mathcal{L}_t(\hat{\Pi}_1^0) + (1-w)\mathcal{L}_t(\hat{\Pi}_2^0)$$

and, by the linearity of \mathcal{L}_t , $\Pi_3^t = \mathcal{L}_t(\hat{\Pi}_3^0)$. Thus, the von Neumann recipe gives $\rho_3^t = w\hat{\rho}_1^t + (1-w)\hat{\rho}_2^t$ and, for linear observables, it yields the correct statistics of measurements,

$$\text{Tr}(A\rho_3^t) = w\text{Tr}(A\hat{\rho}_1^t) + (1-w)\text{Tr}(A\hat{\rho}_2^t).$$

It also entails that for an N level system, a full tomography of a heterogeneous preparation requires the measurement of a quorum of only N^2-1 independent linear observables.

Instead, if the evolution map for homogeneous preparations is nonlinear, $\hat{\Pi}^t = \mathcal{N}_t(\hat{\Pi}^0)$, and the statistical mixing weights remain time-invariant, we obtain

$$\Pi_3^t = w \hat{\Pi}_1^t + (1-w) \hat{\Pi}_2^t = w \mathcal{N}_t(\hat{\Pi}_1^0) + (1-w) \mathcal{N}_t(\Pi_2^0)$$

and, in general, due to the nonlinearity of \mathcal{N}_t , this does not satisfy $\Pi_3^t \neq \mathcal{N}_t(\hat{\Pi}_3^0)$. This contradicts the von Neumann prescription, which would assign the density operator $\rho_3^t = w \rho_1^t + (1-w) \rho_2^t$ to preparation Π_3^t and $\rho_3^0 = w \rho_1^0 + (1-w) \rho_2^0$ to preparation Π_3^0 . As noted by Rembieliński and Caban [39], a sufficient condition for a nonlinear map to be no-signaling is convex-quasilinearity. That is, it must always admit a w' , with $0 \leq w' \leq 1$, such that $\rho_3^t = w' \rho_1^t + (1-w') \rho_2^t$. Ferrero *et al.* [38] made a similar observation but left its interpretation as an open problem.

The HG ontological hypothesis alleviates this interpretation issue. Regardless of how a density operator is decomposed, w and $1-w$ no longer represent epistemic ignorance. This is because both pure and mixed density operators have ontic status, meaning they represent homogeneous preparations.

5 Measure-theoretic representation of statistics from mixed ensembles

To resolve the Schrödinger-Park paradox about individual states in QSM without contradicting the successes of QM, Beretta [63] proposed replacing the von Neumann prescription for measurement statistics from mixed ensembles with an unambiguous representation akin to classical statistical mechanics. The recipe assigns to every preparation Π (whether homogeneous or heterogeneous) a normalized *measure* μ_Π defined on the (ontic) quantal state domain \mathcal{P} , which consists of the mathematical objects representing homogeneous preparations (homogeneous ensembles). The normalization condition is

$$\mu_\Pi(\mathcal{P}) = \int_{\mathcal{P}} \mu_\Pi(d\rho) = 1. \quad (3)$$

In usual von Neumann QSM, the quantal state domain \mathcal{P}_{QSM} is the set of all one-dimensional projection operators (*i.e.*, the idempotent density operators) defined on the Hilbert space of the system. Instead, the HG unified theory assumes the broader quantal state domain \mathcal{P}_{HG} consisting of the set of all possible density operators ρ , idempotent and not.

Among the measures defined on \mathcal{P} , the *Dirac measures*, defined as follows, play a crucial role. Let \mathcal{E} denote any subset of \mathcal{P} , then

$$\delta_\rho(\mathcal{E}) = \begin{cases} 1, & \text{if } \rho \in \mathcal{E} \\ 0, & \text{if } \rho \notin \mathcal{E} \end{cases} \quad (4)$$

The support of a Dirac measure $\delta_{\tilde{\rho}}$, *i.e.*, the subset of \mathcal{P} for which it is nonzero, is a single-point coinciding with the state operator $\tilde{\rho}$.

The statistical mixing of preparations, $\Pi_3 = w \Pi_1 + (1-w) \Pi_2$, is represented by the weighted sum of the corresponding measures,

$$\mu_{\Pi_3} = w \mu_{\Pi_1} + (1-w) \mu_{\Pi_2}. \quad (5)$$

This representation removes ambiguities because:

1. No Dirac measure can be decomposed into a weighted sum of different Dirac measures, satisfying von Neumann's definition of homogeneous preparations.
2. Any measure has a unique decomposition into a weighted sum (or integral) of Dirac measures — removing the Schrödinger-Park paradox. This ensures that each member of a heterogeneous ensemble is in some well defined (albeit unknown) individual state.

The expected mean value of any physical observable represented by the (linear or nonlinear) functional g defined on \mathcal{P} , is given for preparation $\Pi_3 = w \Pi_1 + (1-w) \Pi_2$ by

$$\begin{aligned} \overline{g}_{\Pi_3} &= \int_{\mathcal{P}} g(\rho) \mu_{\Pi_3}(d\rho) \\ &= \int_{\mathcal{P}} g(\rho) [w \mu_{\Pi_1}(d\rho) + (1-w) \mu_{\Pi_2}(d\rho)] \\ &= w \int_{\mathcal{P}} g(\rho) \mu_{\Pi_1}(d\rho) + (1-w) \int_{\mathcal{P}} g(\rho) \mu_{\Pi_2}(d\rho). \end{aligned} \quad (6)$$

If the component preparations are homogeneous,

i.e., if $\mu_{\hat{\Pi}_1} = \delta_{\rho_1}$ and $\mu_{\hat{\Pi}_2} = \delta_{\rho_2}$, then

$$\begin{aligned}\overline{\langle g \rangle}_{\Pi_3} &= w \int_{\mathcal{P}} g(\rho) \delta_{\rho_1}(\mathrm{d}\rho) + (1-w) \int_{\mathcal{P}} g(\rho) \delta_{\rho_2}(\mathrm{d}\rho) \\ &= w g(\rho_1) + (1-w) g(\rho_2) \\ &= w \langle g \rangle_{\hat{\Pi}_1} + (1-w) \langle g \rangle_{\hat{\Pi}_2}.\end{aligned}\quad (7)$$

For example, for $\mathcal{P} = \mathcal{P}_{\text{HG}}$ and $g(\rho) = -k_{\text{B}} \text{Tr}(\rho \ln \rho)$, this gives the ‘proper’ expected value of measurements of the von Neumann entropy, *i.e.*, the weighted sum of the entropies of the component homogeneous sub-ensembles.

It is noteworthy that for observables represented by linear functionals on \mathcal{P} , such as $g(\rho) = \text{Tr}(G\rho)$, we have

$$\begin{aligned}\overline{\langle g \rangle}_{\Pi_3} &= w \text{Tr}(G \rho_{\hat{\Pi}_1}) + (1-w) \text{Tr}(G \rho_{\hat{\Pi}_2}) \\ &= \text{Tr}(G [w \rho_{\hat{\Pi}_1} + (1-w) \rho_{\hat{\Pi}_2}]) \\ &= \text{Tr}(G W_{\Pi_3}) = g(W_{\Pi_3}),\end{aligned}\quad (8)$$

where W_{Π_3} is the usual von Neumann statistical operator, defined by the weighted sum of the density operators representing the component homogeneous sub-ensembles,

$$W_{\Pi_3} = w \rho_{\hat{\Pi}_1} + (1-w) \rho_{\hat{\Pi}_2}.\quad (9)$$

W_{Π_3} is clearly an element of the convex completion, $\mathcal{C}(\mathcal{P})$, of the quantal state domain \mathcal{P} . Notice that in general $\mathcal{C}(\mathcal{P}_{\text{QSM}}) = \mathcal{C}(\mathcal{P}_{\text{HG}}) = \mathcal{P}_{\text{HG}}$, and for a qubit \mathcal{P}_{QSM} and \mathcal{P}_{HG} map to the Bloch sphere and the Bloch ball, respectively. Evaluation of the linear functional $g(\cdot) = \text{Tr}(G\cdot)$ at W_{Π_3} provides the correct expected mean value of measurements of the corresponding observable on the heterogeneous ensemble.

But W_{Π_3} does not fully describe the heterogeneous preparation Π_3 , because different preparations can yield the same linear tomography. In other words, the linear tomography — obtained by measuring the expected mean values $\overline{\langle q_j \rangle}$ of a quorum of $N^2 - 1$ independent linear observables Q_j and solving $\overline{\langle q_j \rangle} = \text{Tr}(Q_j W)$ for W — is insufficient to characterize a preparation. A decomposition of W into different weighted sums of density operators has no meaning in this theory, but to emphasize that different heterogeneous preparations may result in the same linear tomography. Even within orthodox QSM ($\mathcal{P} = \mathcal{P}_{\text{QSM}}$), resolving the intrinsic quantum probabilities of homogeneous preparations from the extrinsic uncertainties of mixing requires additional independent information beyond linear tomography.

The measure-theoretic description of preparations enables a statistical quantum theory where both linear and nonlinear functionals of the density operator correspond to independently and directly measurable properties of a quantum system, such as entropy, ergotropy, and adiabatic availability. A nonlinear evolution equation for the density operators in \mathcal{P} , *i.e.*, for the homogeneous preparations, may provide additional nonlinear observables by measuring linear and nonlinear observables at different times. As already realized by Park [56], nonlinearity holds the promise to preserve and reintegrate the notion of individual state into quantum theory.

6 Local perception operators (LPOs)

In linear QM, the system’s composition is specified by declaring:

1. The Hilbert space structure as the direct product $\mathcal{H} = \bigotimes_{J=1}^M \mathcal{H}_J$ of the subspaces of the M component subsystems.
2. The overall Hamiltonian operator $H = \sum_{J=1}^M H_J \otimes I_{\bar{J}} + V$ where H_J (on \mathcal{H}_J) is the local Hamiltonian of the J -th subsystem, $I_{\bar{J}}$ the identity on the direct product $\mathcal{H}_{\bar{J}} = \bigotimes_{K \neq J} \mathcal{H}_K$ of all the other subspaces, and V (on \mathcal{H}) is the interaction Hamiltonian.

The linear von Neumann law of evolution, $\dot{\rho} = -i[H, \rho]/\hbar$, has a universal structure and involves local evolutions through partial tracing,

$$\dot{\rho}_J = -\frac{i}{\hbar} [H_J, \rho_J] - \frac{i}{\hbar} \text{Tr}_{\bar{J}}([V, \rho]).\quad (10)$$

Thus, we recover the universal law $\dot{\rho}_J = -i[H_J, \rho_J]/\hbar$ for the local density operator $\rho_J = \text{Tr}_{\bar{J}}(\rho)$ if the subsystem J does not interact with the others (*i.e.*, if $V = I_J \otimes V_{\bar{J}}$).

Instead, a fully nonlinear evolution equation for the density operator cannot have a universal structure because the subdivision into subsystems must be explicitly embedded into the structure of the dynamical law (see [25] for more on this). A different subdivision requires a different equation of motion. The complex structure of the SEA evolution law reflects the cost of abandoning linearity but ensures compatibility with the crucial constraint that correlations should not build

up, and signaling should not occur between subsystems, other than via the interaction Hamiltonian V through the standard Schrödinger term $-i[H, \rho]/\hbar$ in the evolution law.

Seldom used in composite quantum dynamics but crucial, in our opinion, are the physical observables first introduced in [17] and later (starting with [19]) referred to as ‘local perceptions of global observables’. These are represented by the ‘local perception operators’ (LPO) on \mathcal{H}_J , defined along with their ‘deviation from the local mean value’ operators and covariance functionals as follows:

$$(X)_\rho^J = \text{Tr}_{\bar{J}}[(I_J \otimes \rho_{\bar{J}})X], \quad (11)$$

$$\Delta(X)_\rho^J = (X)_\rho^J - I_J \text{Tr}[\rho_J(X)_\rho^J], \quad (12)$$

$$(X, Y)_\rho^J = \frac{1}{2} \text{Tr}[\rho_J \{ \Delta(X)_\rho^J, \Delta(Y)_\rho^J \}], \quad (13)$$

where $\rho_{\bar{J}} = \text{Tr}_J(\rho)$. For a bipartite system AB , the LPOs $(X)_\rho^A$ (on \mathcal{H}_A) and $(X)_\rho^B$ (on \mathcal{H}_B) are the unique operators that for a given X on \mathcal{H}_{AB} and for all states ρ , satisfy the identity

$$\text{Tr}[\rho_A(X)_\rho^A] = \text{Tr}[(\rho_A \otimes \rho_B)X] = \text{Tr}[\rho_B(X)_\rho^B]. \quad (14)$$

This confirms that they encapsulate all the information that A and B can infer about the global observable X by classically sharing their local states.

Operator $(X)_\rho^A$ can be viewed as the projection onto \mathcal{H}_A of the operator X weighted by the local state ρ_B of subsystem B . However, it is a local observable for subsystem A which depends on the overall state ρ and overall observable X . Its local mean value $\text{Tr}[\rho_A(X)_\rho^A]$ differs from the mean value $\text{Tr}(\rho X)$ for the overall system AB , except when A and B are uncorrelated ($\rho = \rho_A \otimes \rho_B$). It was dubbed ‘local perception’ because even if B performs a local tomography and sends the measured ρ_B to A by classical communication, the most that A can measure locally about the overall observable X is $(X)_\rho^A$.

The overall energy and entropy of the composite system are locally perceived within subsystem J through the operators $(H)_\rho^J$ and $(S)_\rho^J$ defined on \mathcal{H}_J by Eq. (11), respectively with $X = H$, the overall Hamiltonian, and $X = S(\rho)$, the overall (non-negative) entropy operator defined by

$$S(\rho) = -k_B \text{Bln}(\rho), \quad (15)$$

where we define the discontinuous log function

$$\text{Bln}(x) = \begin{cases} \ln(x), & \text{for } 0 < x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

Note that the ‘locally perceived overall entropy’ operator

$$(S)_\rho^J = -k_B \text{Tr}_{\bar{J}}[(I_J \otimes \rho_{\bar{J}})\text{Bln}(\rho)], \quad (17)$$

is different from the ‘local entropy’ operator

$$S(\rho_J) = -k_B \text{Bln}(\rho_J). \quad (18)$$

Its local mean value $\text{Tr}[\rho_J(S)_\rho^J] = -k_B \text{Tr}[(\rho_J \otimes \rho_{\bar{J}})\text{Bln}(\rho)]$ is different from the local entropy $\text{Tr}[\rho_J S(\rho_J)] = -k_B \text{Tr}[\rho_J \ln(\rho_J)]$. Only when $\rho = \rho_J \otimes \rho_{\bar{J}}$ they are related by

$$\begin{aligned} \text{Tr}[\rho_J(S)_\rho^J] &= \text{Tr}[\rho_J S(\rho_J)] + \text{Tr}[\rho_{\bar{J}} S(\rho_{\bar{J}})] \\ &= -k_B \text{Tr}[\rho \ln(\rho)]. \end{aligned} \quad (19)$$

Likewise, the ‘locally perceived overall Hamiltonian’ operator $(H)_\rho^J$ differs from the ‘local Hamiltonian’ operator H_J . Its local mean value $\text{Tr}[\rho_J(H)_\rho^J] = \text{Tr}[(\rho_J \otimes \rho_{\bar{J}})H]$ is different from the local mean energy $\text{Tr}(\rho_J H_J)$, and only when $V = I_J \otimes V_{\bar{J}}$ they are related by

$$\text{Tr}[\rho_J(H)_\rho^J] = \text{Tr}(\rho_J H_J) + \text{Tr}(\rho_{\bar{J}} H_{\bar{J}}) = \text{Tr}(\rho H). \quad (20)$$

However, it is noteworthy that when the overall observable X is ‘separable for subsystem J ,’ in the sense that $X = X_J \otimes I_{\bar{J}} + I_J \otimes X_{\bar{J}}$ then, even if $\rho \neq \rho_J \otimes \rho_{\bar{J}}$, the deviations and covariances reduce to their local versions,

$$\Delta(X)_\rho^J = \Delta X_J = X_J - I_J \text{Tr}[\rho_J X_J], \quad (21)$$

$$(X, Y)_\rho^J = \frac{1}{2} \text{Tr}[\rho_J \{ \Delta X_J, \Delta Y_J \}]. \quad (22)$$

In general, the mean interaction energy, $\text{Tr}(\rho_J V_{J\bar{J}})$, and the mutual information, $\text{Tr}(\rho_J \mu_{J\bar{J}})$, between subsystem J and the rest of the system, \bar{J} , are given by the respective mean values of the following global operators

$$V_{J\bar{J}} = H - H_J \otimes I_{\bar{J}} - I_J \otimes H_{\bar{J}}, \quad (23)$$

$$\mu_{J\bar{J}} = \text{Bln}(\rho) - \text{Bln}(\rho_J) \otimes I_{\bar{J}} - I_J \otimes \text{Bln}(\rho_{\bar{J}}), \quad (24)$$

whose LPOs satisfy the identities

$$\begin{aligned} \text{Tr}[\rho_J(H)_\rho^J] - \text{Tr}[\rho_J(V_{J\bar{J}})_\rho^J] &= \text{Tr}(\rho_J H_J) + \text{Tr}(\rho_{\bar{J}} H_{\bar{J}}) \\ &= \text{Tr}(\rho H) - \text{Tr}(\rho_J V_{J\bar{J}}), \end{aligned} \quad (25)$$

$$\begin{aligned}
& \text{Tr}[\rho_J(S)_\rho^J] + k_B \text{Tr}[\rho_J(\mu_{J\bar{J}})_\rho^J] \\
&= -k_B \text{Tr}[\rho_J \ln(\rho_J)] - k_B \text{Tr}[\rho_{\bar{J}} \ln(\rho_{\bar{J}})] \\
&= -k_B \text{Tr}[\rho \ln(\rho)] + k_B \text{Tr}(\rho_J \mu_{J\bar{J}}). \quad (26)
\end{aligned}$$

7 No-signaling and LPOs

To formalize the no-signaling definition following [22] as discussed above, we adopt the view that local operations can be acted on a subsystem \bar{J} only by means of a controlled time dependence of its local Hamiltonian operator $H_{\bar{J}}$ or its interaction operator $V_{\bar{J}B}$ with some other subsystem B (possibly a properly modeled environment or heat bath). Hence, we adopt the following

Definition (No-signaling). *If J and \bar{J} are non-interacting, no local aspect of the time evolution of J can be affected by local unitary operations acted on \bar{J} nor by other aspects of the local time evolution of \bar{J} .*

Accordingly, consider a composite AB in state ρ . A local unitary operation on B changes the state to

$$\rho' = (I_A \otimes U_B) \rho (I_A \otimes U_B^\dagger), \quad (27)$$

where U_B is an arbitrary unitary operator ($U_B^\dagger U_B = I_B$). Using the properties of the partial trace, in particular,

$$\text{Tr}_B[(I_A \otimes X_B) Z_{AB}] = \text{Tr}_B[Z_{AB} (I_A \otimes X_B)], \quad (28)$$

$$\text{Tr}_A[(I_A \otimes X_B) Z_{AB} (I_A \otimes Y_B)] = X_B \text{Tr}_A(Z_{AB}) Y_B, \quad (29)$$

we obtain the identities

$$\rho_B = \text{Tr}_A[(I_A \otimes U_B^\dagger) \rho' (I_A \otimes U_B)] = U_B^\dagger \rho'_B U_B, \quad (30)$$

$$\begin{aligned}
\rho'_A &= \text{Tr}_B[(I_A \otimes U_B) \rho (I_A \otimes U_B^\dagger)] \\
&= \text{Tr}_B[(I_A \otimes U_B^\dagger U_B) \rho] \\
&= \text{Tr}_B[(I_A \otimes I_B) \rho] = \rho_A, \quad (31)
\end{aligned}$$

which confirms that a local operation on B does not affect the local state ρ_A of A . This result supports the usual idea [38] that for no-signaling it is sufficient that the dynamical model implies evolutions of local observables that depend only on ρ_A . But it is seldom noted that this is not a necessary condition.

Next, we prove that not only the local reduced state ρ_A but also the LPO $(F(\rho))^A$, of any well-defined nonlinear function $F(\rho)$ of the overall state (such as the entropy function $S(\rho)$ defined earlier) remains unaffected by local unitary operations on B as per Eq. (27). Since the SEA formalism is based on these local perception operators, this result is an important lemma in the proof that SEA is no-signaling.

So, let us apply Eq. (27) to a generic observable represented by a linear operator on \mathcal{H} that we denote by $F(\rho)$, whether it is a nontrivial function of ρ , such as $S(\rho)$, or not a function of ρ , such as H . The corresponding LPO for subsystem A , according to the defining Eq. (11), is given by

$$(F(\rho))^A = \text{Tr}_B[(I_A \otimes \rho_B) F(\rho)]. \quad (32)$$

A function of ρ is defined from its eigen-decomposition by

$$F(\rho) = \Lambda F(D) \Lambda^\dagger = \sum_j F(\lambda_j) |\lambda_j\rangle\langle\lambda_j|,$$

where $\rho = \Lambda D \Lambda^\dagger$, $D = \sum_j \lambda_j |j\rangle\langle j|$, and $\Lambda = \sum_j |\lambda_j\rangle\langle j|$. Since unitary transformations do not alter the eigenvalues,

$$F(\rho') = \Lambda' F(D) \Lambda'^\dagger \text{ where } \Lambda' = (I_A \otimes U_B) \Lambda. \quad (33)$$

Therefore, using Eq. (30) in the last step, we obtain

$$\begin{aligned}
(F(\rho'))^A &= \text{Tr}_B[(I_A \otimes \rho'_B) F(\rho')] \\
&= \text{Tr}_B[(I_A \otimes \rho'_B) (I_A \otimes U_B) \Lambda F(D) \Lambda^\dagger (I_A \otimes U_B^\dagger)] \\
&= \text{Tr}_B[(I_A \otimes U_B^\dagger \rho'_B U_B) \Lambda F(D) \Lambda^\dagger] \\
&= \text{Tr}_B[(I_A \otimes \rho_B) F(\rho)] = (F(\rho))^A. \quad (34)
\end{aligned}$$

This confirms that local unitary operations on B do not affect the LPOs of A . Hence, the proper use of LPOs in a nonlinear evolution equation does not cause signaling issues.

8 The local structure of dissipation in composite-system SEA dynamics

We are now ready to introduce the final but essential ingredient of a general composite-system nonlinear QM or master equation. This involves the system's structure-dependent expressions, which determine how each subsystem contributes separately to the dissipative term in the equation

of motion for the overall state ρ . As discussed above — and recognized in early SEA literature [17, 22, 25] — the composite-system nonlinear evolution must explicitly reflect the internal structure of the system. This requires declaring which subsystems are to be protected from nonphysical effects such as signaling, exchange of energy, or build-up of correlations between non-interacting subsystems. Using the notation introduced earlier, the structure proposed in [17, 25] for the dissipative term, which supplements the usual Hamiltonian term, is given by

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] - \sum_{J=1}^M \{\mathcal{D}_\rho^J, \rho_J\} \otimes \rho_{\bar{J}}, \quad (35)$$

where the ‘local dissipation operators’ \mathcal{D}_ρ^J (on \mathcal{H}_J) may be nonlinear functions of the local observables of J , the reduced state ρ_J , and the local perception operators of overall observables. For the dissipative term to preserve $\text{Tr}(\rho)$, operators $\{\mathcal{D}_\rho^J, \rho_J\}$ must be traceless. To preserve $\text{Tr}(\rho H)$ [and possibly other conserved properties or charges, $\text{Tr}(\rho C_k)$], operators $\{\mathcal{D}_\rho^J, \rho_J\} (H)_\rho^J$ [and $\{\mathcal{D}_\rho^J, \rho_J\} (C_k)_\rho^J$] must also be traceless. The rate of change of the overall system entropy $s(\rho) = -k_B \text{Tr}[\rho \ln(\rho)]$ is

$$\frac{ds(\rho)}{dt} = - \sum_{J=1}^M \text{Tr}[\{\mathcal{D}_\rho^J, \rho_J\} (S)_\rho^J]. \quad (36)$$

The local nonlinear evolution of subsystem J is obtained by partial tracing over $\mathcal{H}_{\bar{J}}$, *i.e.*, in general,

$$\frac{d\rho_J}{dt} = -\frac{i}{\hbar}[H_J, \rho_J] - \frac{i}{\hbar} \text{Tr}_{\bar{J}}([V, \rho]) - \{\mathcal{D}_\rho^J, \rho_J\}, \quad (37)$$

where we recall that the second term in the RHS can be put, for weak interactions and under well-known assumptions, in Kossakowski-Lindblad form.

Before introducing the SEA assumption, as promised after Eq. (2), we note that for all possible choices of \mathcal{D}_ρ^J , Eq. (35) defines a broad class of no-signaling nonlinear evolution equations. These form a broader class of nonlinear laws which are not restricted by the sufficient but not necessary condition that $d\rho_J/dt$ be a function of ρ_J only.

Finally, one way to introduce the SEA assumption in the spirit of the fourth law of ther-

modynamics [32], *i.e.*, to implement the maximum entropy production principle (MEPP) in the present context [31], is to employ a variational principle.

9 General composite system version of the SEA variational principle

For the purpose of this section, we adopt the formalism developed for this context in [17, 20, 25]. To easily impose the constraints of preservation of non-negativity and self-adjointness of ρ during its time evolution, we define the generalized square root of ρ_J , $\gamma_J(t) = \sqrt{\rho_J(t)} U$ (U any unitary operator) so that,

$$\rho_J = \gamma_J \gamma_J^\dagger. \quad (38)$$

The dissipative term in Eq. (37) rewritten as

$$-\{\mathcal{D}_\rho^J, \rho_J\} = \dot{\gamma}_J^d \gamma_J^\dagger + \gamma_J \dot{\gamma}_J^{d\dagger} \quad (39)$$

$$\text{with } \dot{\gamma}_J^d = -\mathcal{D}_\rho^J \gamma_J. \quad (40)$$

Next, on the set $\mathcal{L}(\mathcal{H}_J)$ of linear operators on \mathcal{H}_J we define the real inner product $(\cdot|\cdot)$

$$(X|Y) = \text{Tr}(X^\dagger Y + Y^\dagger X)/2, \quad (41)$$

so that the unit trace condition for ρ_J rewrites as $(\gamma_J|\gamma_J) = 1$, implying that the γ_J ’s lie on the unit sphere in $\mathcal{L}(\mathcal{H}_J)$. Their time evolutions $\gamma_J(t)$ follow trajectories on this sphere. Along these trajectories, we can express the distance traveled between t and $t + dt$ as

$$d\ell_J = \sqrt{(\dot{\gamma}_J|\hat{G}_J(\gamma_J)|\dot{\gamma}_J)} dt, \quad (42)$$

where $\hat{G}_J(\gamma_J)$ is some real, dimensionless, symmetric, and positive-definite operator on $\mathcal{L}(\mathcal{H}_J)$ (superoperator on \mathcal{H}) that plays the role of a local metric tensor field (and may be a nonlinear function of γ_J).

The rates of change of the overall system entropy $s(\rho)$, Eq. 36, and of the overall system mean value of conserved properties $c_k(\rho) = \text{Tr}(\rho C_k)$, where $[C_k, H] = 0$, can be rewritten as

$$\frac{ds(\rho)}{dt} = \sum_{J=1}^M \dot{s}|_J \quad \dot{s}|_J = \left(2(S)_\rho^J \gamma_J \middle| \dot{\gamma}_J^d\right), \quad (43)$$

$$\frac{dc_k(\rho)}{dt} = \sum_{J=1}^M \dot{c}_k|_J \quad \dot{c}_k|_J = \left(2(C_k)_\rho^J \gamma_J \middle| \dot{\gamma}_J^d\right), \quad (44)$$

exhibiting additive contributions from the sub-systems.

Finally we state the variational principle that leads to expressions for the $\dot{\gamma}_J^d$'s and the \mathcal{D}_ρ^J 's that define the composite-system version of the SEA equation of motion. The time evolution ensures that the “direction of change” of the local trajectory $\gamma_J(t)$, influenced by the dissipative part of the dynamics, maximizes the local contribution $\dot{s}|_J$ to the overall system's entropy production rate. This occurs under the constraints $\dot{c}_k|_J = 0$ that guarantee no local contribution to the rates of change of the global constants of the motion. Introducing Lagrange multipliers ϑ_k^J and τ_J for the constraints, the $\dot{\gamma}_J^d$'s are found by solving the maximization problem

$$\max_{\dot{\gamma}_J^d} \Upsilon_J = \dot{s}|_J - \sum_k \vartheta_k^J \dot{c}_k|_J - \frac{k_B \tau_J}{2} \left(\dot{\gamma}_J^d | \hat{G}_J | \dot{\gamma}_J^d \right), \quad (45)$$

where the last constraint corresponds to the condition $(d\ell_J^d/dt)^2 = \text{const}$, necessary for maximizing with respect to direction only (see [31] for more details). Taking the variational derivative of Υ_J with respect to $|\dot{\gamma}_J^d\rangle$ and setting it equal to zero, we obtain

$$\frac{\delta \Upsilon_J}{|\delta \dot{\gamma}_J^d\rangle} = \left| 2(M)_\rho^J \gamma_J \right\rangle - k_B \tau_J \hat{G}_J | \dot{\gamma}_J^d \rangle = 0, \quad (46)$$

where we used the identity $\langle X | \hat{G}_J = \hat{G}_J | X \rangle$ following from the symmetry of \hat{G}_J . We define (following [24, 70]) the “locally perceived non-equilibrium Massieu operator”

$$(M)_\rho^J = (S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J. \quad (47)$$

Eq. (46) yields

$$| \dot{\gamma}_J^d \rangle = \frac{1}{k_B \tau_J} \hat{G}_J^{-1} | 2(M)_\rho^J \gamma_J \rangle, \quad (48)$$

where the Lagrange multipliers ϑ_k^J (implicit in $(M)_\rho^J$) are the solution of the system of equations, obtained by substituting Eq. (48) into the conservation constraints,

$$\left((C_\ell)_\rho^J \gamma_J | \hat{G}_J^{-1} | (M)_\rho^J \gamma_J \right) = 0 \quad \forall \ell. \quad (49)$$

This system can be solved explicitly for the ϑ_k^J 's using Cramer's rule, to obtain convenient expressions for the $\dot{\gamma}_J^d$'s as ratios of determinants (as

in the original formulations). The ϑ_k^J 's are non-linear functionals of ρ that may be interpreted as “local non-equilibrium entropic potentials” conjugated with the conserved properties. For example, for $C_2 = H$, the Lagrange multiplier ϑ_2^J plays the role of “local non-equilibrium inverse temperature” conjugated with the locally perceived energy, and for the stable equilibrium states of the SEA dynamics it coincides with the thermodynamic inverse temperature $k_B \beta_J$ (see below).

Similarly to what done in [31] for a non-composite system, we define the “local non-equilibrium affinity” operators

$$|\Lambda_J\rangle = \hat{G}_J^{-1/2} | 2(M)_\rho^J \gamma_J \rangle, \quad (50)$$

so that the overall rate of entropy production becomes

$$\frac{ds(\rho)}{dt} = \sum_{J=1}^M \frac{(\Lambda_J | \Lambda_J)}{k_B \tau_J}. \quad (51)$$

$(\Lambda_J | \Lambda_J)$ is the norm of $2(M)_\rho^J \gamma_J$ with respect to the metric \hat{G}_J^{-1} and may be interpreted as the “degree of disequilibrium” of subsystem J . Hence, the necessary and sufficient condition for the overall state to be locally non-dissipative (no contribution to the overall entropy production from subsystem J) is that operator $2(M)_\rho^J \gamma_J$ vanishes.

However, in order for the equation of motion (37) to result independent of the unitary operators U used (in $\gamma_J = \sqrt{\rho_J} U$) to define the generalized square roots of the ρ_J 's, we further restrict the choice of the metric superoperator \hat{G}_J . We assume that $\hat{G}_J = L_J^{-1} \hat{I}_J$ with L_J some strictly positive, hermitian operator on \mathcal{H}_J , possibly a non-linear function of ρ_J , so that $\hat{G}_J | X \rangle = | L_J^{-1} X \rangle$, $\hat{G}_J^{-1} | X \rangle = | L_J X \rangle$,

$$\langle X \gamma_J | \hat{G}_J^{-1} | Y \gamma_J \rangle = \frac{1}{2} \text{Tr} \left[\rho_J (X^\dagger L_J Y + Y^\dagger L_J X) \right]. \quad (52)$$

Recalling Eqs. (39) and (40), and using Eqs. (48) the dissipative term in Eq. (37) becomes

$$-\{ \mathcal{D}_\rho^J, \rho_J \} = \frac{2}{k_B \tau_J} \left[L_J (M)_\rho^J \rho_J + \rho_J (M)_\rho^J L_J \right], \quad (53)$$

and the system of equations that determines the Lagrange multipliers ϑ_k^J in $(M)_\rho^J$ is

$$\text{Tr} \left(\rho_J \left[(C_\ell)_\rho^J L_J (M)_\rho^J + (M)_\rho^J L_J (C_\ell)_\rho^J \right] \right) = 0 \quad \forall \ell \quad (54)$$

so that the dependence on γ_J is only through the product $\gamma_J \gamma_J^\dagger$, *i.e.*, the local state operator ρ_J .

The metric operator $L_J^{-1}/k_B\tau_J$ plays a role analogous to the symmetric thermal conductivity tensor \hat{k} in heat transfer theory. In that context, \hat{k} defines the general near-equilibrium linear relationship, $|q''\rangle = -\hat{k}|\nabla T\rangle$, between the heat flux vector q'' and the conjugated “degree of disequilibrium” vector, *i.e.*, the temperature gradient ∇T . Here, Eq. (53) expresses a more general linear relationship between the local evolution operator $\hat{\gamma}_J^d$ and the non-equilibrium Massieu operator, $(S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J$. This relation is more general as it holds not only near equilibrium, but also anywhere far from equilibrium. In the present quantum modeling context, it represents the nonlinear SEA extension into the far-non-equilibrium domain of Onsager’s linear near-equilibrium theory, with reciprocity naturally embedded through the symmetry of any metric. For example, if L_J commutes with the local Hamiltonian H_J , its eigenvalues can assign different relaxation times to the local energy levels, capturing their uneven contributions to the rate of local energy redistribution described by the SEA dissipator \mathcal{D}_ρ^J .

Like in heat transfer the conductivity tensor for an isotropic material is $\hat{k} = k\hat{I}$, here an analogous simplification is obtained when $L_J = I_J$, the identity operator on \mathcal{H}_J . This corresponds to assuming a uniform Fisher-Rao metric.

Another noteworthy observation is that operators C_k , to be global constants of the motion (or “charges”), must commute with the global Hamiltonian operator H , but they need not commute with each other. Therefore, the SEA formalism may find application also in the framework of quantum thermodynamic resource theories that contemplate “non-commuting charges” and “non-Abelian thermal states” as discussed in Yunger Halpern *et al.* [11] and Murthy *et al.* [71].

10 Simplest composite-system SEA equation of motion

For simplicity, we proceed by assuming: 1) a uniform Fisher-Rao metric, with $\hat{G}_J = \hat{I}_J$, so that $\hat{G}_J|X\rangle = |X\rangle$ and $L_J = I_J$; and 2) only two global conserved properties: $c_1(\rho) = \text{Tr}(\rho I)$ (the normalization condition) and $c_2(\rho) = \text{Tr}(\rho H)$

(mean energy conservation).

The first assumption allows to obtain

$$\mathcal{D}_\rho^J = -\frac{2}{k_B\tau_J} \left[(S)_\rho^J - \sum_k \vartheta_k^J (C_k)_\rho^J \right]. \quad (55)$$

and the system of Eqs. (54) reduces to

$$\begin{aligned} & \sum_k \vartheta_k^J \text{Tr} \left[\rho_J \left\{ (C_\ell)_\rho^J, (C_k)_\rho^J \right\} \right] \\ &= \text{Tr} \left[\rho_J \left\{ (S)_\rho^J, (C_\ell)_\rho^J \right\} \right]. \end{aligned} \quad (56)$$

The second assumption, $C_1 = I$ and $C_2 = H$, together with definitions (12) and (13), allows to write the SEA dissipators in the compact forms

$$\begin{aligned} -\{\mathcal{D}_\rho^J, \rho_J\} &= \frac{2}{k_B\tau_J} \{\Delta(M)_\rho^J, \rho_J\} \\ &= \frac{2}{k_B\tau_J} \frac{\begin{vmatrix} \{\Delta(S)_\rho^J, \rho_J\} & \{\Delta(H)_\rho^J, \rho_J\} \\ (H, S)_\rho^J & (H, H)_\rho^J \end{vmatrix}}{(H, H)_\rho^J}, \end{aligned} \quad (57)$$

and the local non-equilibrium inverse temperatures conjugated with the locally perceived energy as

$$\vartheta_2^J = \frac{(H, S)_\rho^J}{(H, H)_\rho^J}. \quad (58)$$

As a result, the local evolution of each subsystem J is along the direction of steepest ascent of the locally perceived overall composite-system entropy, compatible with the conservation of the locally perceived overall composite-system energy and the conditions of unit-trace and non-negativity of the overall density operator ρ . The rate of entropy production may be expressed as

$$\begin{aligned} \frac{ds(\rho)}{dt} &= \sum_{J=1}^M \frac{4}{k_B\tau_J} (M, M)_\rho^J \\ &= \sum_{J=1}^M \frac{4}{k_B\tau_J} \frac{\begin{vmatrix} (S, S)_\rho^J & (H, S)_\rho^J \\ (H, S)_\rho^J & (H, H)_\rho^J \end{vmatrix}}{(H, H)_\rho^J}, \end{aligned} \quad (59)$$

showing clearly that it is non-negative since the numerators in the summation are Gram determinants. When the J 'th term in the sum vanishes we say that the local state ρ_J is “non-dissipative.” If all local states are non-dissipative, then so is the overall state ρ . A non-dissipative ρ represents an equilibrium state if it commutes with H , otherwise it belongs to a unitary limit cycle of the dynamics.

For ρ to be non-dissipative there must exist β_J 's such, that for every J ,

$$\rho_J \Delta(\text{Bln}(\rho))_\rho^J = -\beta_J \rho_J \Delta(H)_\rho^J, \quad (60)$$

or, equivalently,

$$\rho_J \Delta \left[\text{Bln}(\rho_J) + \beta_J H_J + \beta_J (V_{J\bar{J}})_\rho^J + (\mu_{J\bar{J}})_\rho^J \right] = 0, \quad (61)$$

where $V_{J\bar{J}}$ is the Hamiltonian interaction operator defined in Eq. (23) and $\mu_{J\bar{J}}$ the mutual information operator defined in Eq. (24). Clearly, for $V_{J\bar{J}} = 0$ and $\mu_{J\bar{J}} = 0$, Eq. (61) is satisfied by the Gibbs states

$$\rho_J = \exp(-\beta_J H_J) / \text{Tr}[\exp(-\beta_J H_J)]. \quad (62)$$

Section 11.3 provides a numerical illustration of local time evolutions that converge to limit cycles obeying Eq. (61). These examples also illustrate another noteworthy possibility. Specifically, without violating the no-signaling condition nor the second-law principle of global entropy non-decrease — which is always guaranteed by virtue of Eq. (59) and the properties of Gram determinants — it is possible that the local entropy of a subsystem be decreasing during part of the time evolution. Indeed, for correlated states, the non-decrease of local entropy is not a second-law requirement. We hope that the observation that a local entropy decrease may be thermodynamically consistent in the presence of strong entanglement will stimulate the design of experimental verifications as well as foundational discussions in the framework of quantum thermodynamic resource theories, even beyond and outside the SEA framework.

Regarding no-signaling, we note that:

1. if subsystem J is non-interacting, $V_{J\bar{J}} = 0$, then

$$\Delta(H)_\rho^J = H_J - I_J \text{Tr}(\rho_J H_J) = \Delta H_J$$

and

$$(H, H)_\rho^J = \text{Tr}[\rho_J (\Delta H_J)^2]$$

depend only on the local H_J and ρ_J ;

2. if J is uncorrelated, $\mu_{J\bar{J}} = 0$, then

$$\Delta(\text{Bln}(\rho))_\rho^J = \text{Bln}(\rho_J) - I_J \text{Tr}(\rho_J \ln(\rho_J))$$

and

$$(\text{Bln}(\rho), \text{Bln}(\rho))_\rho^J = \text{Tr}[\rho_J (\ln(\rho_J))^2]$$

depend only on the local ρ_J .

Therefore, it is only when J is both non-interacting and uncorrelated that its local dissipation operator \mathcal{D}_ρ^J depends only on the local H_J and ρ_J . In this case the local equation of motion Eq. (37) along with \mathcal{D}_ρ^J given by Eq. (57) reduces exactly to the non-composite system version of SEA evolution [16]. Instead, if J is either interacting or correlated, \mathcal{D}_ρ^J and, therefore, the local nonlinear SEA evolution, according to Eqs. (37) and (57), is determined not only by the local H_J and ρ_J , but also by the local perceptions of the overall Hamiltonian operator H and/or the overall entropy operator $\text{Bln}(\rho)$, nonetheless without violating the no-signaling condition.

To prove no-signaling, assume that subsystem J is correlated but not interacting with any of the subsystems in \bar{J} . Now switch on an arbitrary interaction that may involve the subsystems in \bar{J} , but not subsystem J , so that the only change in the overall Hamiltonian H is the term $H_{\bar{J}}$ which changes to $H_{\bar{J}}'$. Within J , the locally perceived deviation operators are $\Delta(H)_\rho^J$ and $\Delta(\text{Bln}(\rho))_\rho^J$, and hence the SEA dissipator \mathcal{D}_ρ^J as well as all the terms in the RHS of Eq. (37) are not modified by such change. Therefore, acting within \bar{J} makes it impossible to affect the time evolution of ρ_J and any local observable of J .

We also mention the following in the spirit of non-linear evolutions. In the case of a general open quantum dynamics, the requirement that a map $\mathcal{W}_t(\rho)$ be completely positive and trace preserving (CPTP) is quite restrictive [72–74]. In fact, the reduced dynamics of the subsystem in interaction with the environment needs not be CPTP [75–77]. Here, we define CPTP as follows. If $\mathcal{M}(\rho_s)$ is a map acting on a subsystem in state ρ_s , which interacts with an initially uncorrelated environment in state ρ_E , and is positive and trace preserving — meaning it preserves the trace and ensures the evolved matrix remains semi-definite (maintaining probability conservation)— then $\mathcal{M}(\rho_s)$ is CPTP if and only if the extended map $\Lambda : \mathcal{M}(\rho_s) \otimes I_N$ remains positive for all N 's [72]. As it has been argued in the literature, the CPTP condition is restrictive because along with underlying assumption of Markovianity, it requires preparation of the initial state into a product state of system and environment. But, if there are strong initial correlations between system and environment, or in fact, if the evolution is non-Markovian in nature, then

this requirement fails and we are forced to consider PTP (positive and trace preserving) maps only [78, 79]. Given that we are dealing with a theory which does not restrict itself to Markovian evolution, and that there is no imposition of weak interaction between the subsystems, SEA being PTP satisfies the requirements for a suitable nonlinear model of quantum thermodynamics prohibiting signaling.

11 Non-interacting qubits

In extremal cases, Eq. (61) shows that even if the subsystems are entangled, and therefore the local states ρ_J are mixed, operators \mathcal{D}_ρ^J may vanish. Eqs. (35) and (37) reduce to the standard Schrödinger equation and the trajectory in state space is a limit cycle of the SEA dynamics. A noteworthy particular case is when the overall system is in a pure state. Then, $\text{Bln}(\rho) = 0$ and standard unitary evolutions of pure states emerge as limit cycles of the nonlinear SEA dynamics. In this section we discuss a few less trivial examples, to numerically illustrate, some general features of Eq. (35) with \mathcal{D}_ρ^J given by Eq. (57).

We consider examples of mixed and correlated initial states of a two-qubit composite AB that belong to the special class

$$\begin{aligned} \rho &= \frac{1}{4} \left[\mathbb{I}_4 + \sum_{j=\{x,y,z\}} (a_j \sigma_j \otimes \mathbb{I}_2 + b_j \mathbb{I}_2 \otimes \sigma_j + c_j \sigma_j \otimes \sigma_j) \right] \\ &= \frac{1}{4} \begin{pmatrix} 1+a_z+b_z+c_z & b_x-ib_y & a_x-ia_y & c_x-c_y \\ b_x+ib_y & 1+a_z-b_z-c_z & c_x+c_y & a_x-ia_y \\ a_x+ia_y & c_x+c_y & 1-a_z+b_z-c_z & b_x-ib_y \\ c_x-c_y & a_x+ia_y & b_x+ib_y & 1-a_z-b_z+c_z \end{pmatrix}, \end{aligned} \quad (63)$$

with real a_j 's, b_j 's, c_j 's, such that $a^2 = \mathbf{a} \cdot \mathbf{a} \leq 1$, $b^2 = \mathbf{b} \cdot \mathbf{b} \leq 1$, $c^2 = \mathbf{c} \cdot \mathbf{c} \leq 3 - a^2 - b^2$, plus other conditions necessary for non-negativity (see e.g., [80–82]). We will denote the eigenvalues of ρ by λ_j and, for shorthand, define

$$\eta_j = -k_B \text{Bln}(\lambda_j). \quad (64)$$

We further assume that the two qubits are non-interacting and have local Hamiltonian operators given by $H_J = \omega_J \mathbf{h}^J \cdot \boldsymbol{\sigma}_J$ where $\boldsymbol{\sigma}_J$ denotes the 3-vector formed by the local Pauli operators of subsystem J , and \mathbf{h}^J is the local Hamiltonian

unit 3-vector, so that

$$\begin{aligned} H &= H_A \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes H_B \\ &= \omega_A \mathbf{h}^A \cdot \boldsymbol{\sigma}_A \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \omega_B \mathbf{h}^B \cdot \boldsymbol{\sigma}_B, \\ (H)_\rho^A &= H_A + \mathbb{I}_2 \omega_B \mathbf{h}^B \cdot \mathbf{b}, \\ (H)_\rho^B &= \mathbb{I}_2 \omega_A \mathbf{h}^A \cdot \mathbf{a} + H_B \\ \Delta(H)^A &= \Delta H_A, \quad \Delta(H)^B = \Delta H_B, \\ (H, H)_\rho^A &= [1 - (\mathbf{h}^A \cdot \mathbf{a})^2] \omega_A^2, \\ (H, H)_\rho^B &= [1 - (\mathbf{h}^B \cdot \mathbf{b})^2] \omega_B^2, \end{aligned} \quad (65)$$

11.1 Bell diagonal states

Eq. (63) gives Bell diagonal states [83, 84] (BDS) if $a_j = b_j = 0$ for all j 's (and Werner states [85] if in addition $c_j = 4w/3 - 1$ for all j 's),

$$\begin{aligned} \rho^{\text{Bell}} &= \frac{1}{4} \begin{pmatrix} 1+c_z & 0 & 0 & c_x-c_y \\ 0 & 1-c_z & c_x+c_y & 0 \\ 0 & c_x+c_y & 1-c_z & 0 \\ c_x-c_y & 0 & 0 & 1+c_z \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \lambda_2+\lambda_3 & 0 & 0 & \lambda_3-\lambda_2 \\ 0 & \lambda_1+\lambda_4 & \lambda_4-\lambda_1 & 0 \\ 0 & \lambda_4-\lambda_1 & \lambda_1+\lambda_4 & 0 \\ \lambda_3-\lambda_2 & 0 & 0 & \lambda_2+\lambda_3 \end{pmatrix}, \\ &= \rho_A \otimes \rho_B + \sum_{j=\{x,y,z\}} c_j \sigma_j \otimes \sigma_j \end{aligned} \quad (66)$$

whose eigen-decomposition may be written as

$$\begin{aligned} \rho^{\text{Bell}} &= U^{\text{Bell}} \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) (U^{\text{Bell}})^\dagger, \\ U^{\text{Bell}} &= \frac{\sigma_x \otimes \mathbb{I}_2 - \sigma_z \otimes \sigma_x}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\ \lambda_1 &= \frac{1}{4} (1 - c_x - c_y - c_z), \quad \lambda_2 = \frac{1}{4} (1 - c_x + c_y + c_z), \\ \lambda_3 &= \frac{1}{4} (1 + c_x - c_y + c_z), \quad \lambda_4 = \frac{1}{4} (1 + c_x + c_y - c_z). \end{aligned} \quad (67)$$

The overall entropy operator, Eq. (15), becomes

$$\begin{aligned} S(\rho^{\text{Bell}}) &= U^{\text{Bell}} [\text{diag}(\eta_1, \eta_2, \eta_3, \eta_4)] (U^{\text{Bell}})^\dagger \\ &= \frac{1}{2} \begin{pmatrix} \eta_2+\eta_3 & 0 & 0 & \eta_3-\eta_2 \\ 0 & \eta_1+\eta_4 & \eta_4-\eta_1 & 0 \\ 0 & \eta_4-\eta_1 & \eta_1+\eta_4 & 0 \\ \eta_3-\eta_2 & 0 & 0 & \eta_2+\eta_3 \end{pmatrix}, \end{aligned} \quad (68)$$

and the local operators of the SEA formalism

$$\begin{aligned} \rho_A &= \rho_B = \frac{1}{2} \mathbb{I}_2, \\ (\text{Bln}(\rho))^A &= (\text{Bln}(\rho))^B = -\frac{q}{4k_B} \mathbb{I}_2, \\ \{(\text{Bln}(\rho))^A, \rho_A\} &= \{(\text{Bln}(\rho))^B, \rho_B\} = -\frac{q}{4k_B} \mathbb{I}_2, \\ \{\Delta(\text{Bln}(\rho))^A, \rho_A\} &= \{\Delta(\text{Bln}(\rho))^B, \rho_B\} = 0, \\ (H, H)_\rho^A &= \omega_A^2, \quad (H, H)_\rho^B = \omega_B^2, \\ (H, \text{Bln}(\rho))_\rho^A &= (H, \text{Bln}(\rho))_\rho^B = 0. \end{aligned} \quad (69)$$

where q is defined as

$$q = \eta_1 + \eta_2 + \eta_3 + \eta_4. \quad (70)$$

Therefore, somewhat surprisingly, we find that

$$-\{\mathcal{D}_\rho^A, \rho_A\} = -\{\mathcal{D}_\rho^B, \rho_B\} = 0, \quad (71)$$

i.e., the Bell diagonal states are non-dissipative limit cycles of the nonlinear SEA dynamics under any Hamiltonian. But most neighboring and other states in the class defined by Eq. (63) are dissipative, as we see in the following examples.

11.2 Separable, but correlated mixed states

For a simple example of correlated but separable mixed states, assume Eq. (63) with $a_x = a$, $b_z = b$, and $a_y = a_z = b_x = b_y = c_x = c_y = c_z = 0$, so that

$$\begin{aligned} \rho &= \frac{1}{4} \begin{pmatrix} 1+b & 0 & a & 0 \\ 0 & 1-b & 0 & a \\ a & 0 & 1+b & 0 \\ 0 & a & 0 & 1-b \end{pmatrix} \\ &= \rho_A \otimes \rho_B - \frac{1}{4} ab \sigma_x \otimes \sigma_z, \\ \lambda_1 &= \frac{1}{4}(1-a-b), \quad \lambda_2 = \frac{1}{4}(1-a+b), \\ \lambda_3 &= \frac{1}{4}(1+a-b), \quad \lambda_4 = \frac{1}{4}(1+a+b). \end{aligned} \quad (72)$$

If the two non-interacting qubits have local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, we find

$$-\{\mathcal{D}_\rho^A, \rho_A\} = \frac{(1-a^2)(bf-g)}{2k_B\tau_A} \sigma_x, \quad (73)$$

$$-\{\mathcal{D}_\rho^B, \rho_B\} = \frac{(1-b^2)(af-p)}{2k_B\tau_B} \sigma_z, \quad (74)$$

where f , g , and p are defined as

$$\begin{aligned} f &= \eta_1 - \eta_2 - \eta_3 + \eta_4, \\ g &= \eta_1 + \eta_2 - \eta_3 - \eta_4, \\ p &= \eta_1 - \eta_2 + \eta_3 - \eta_4, \end{aligned} \quad (75)$$

so that the nonlinear evolution is clearly nontrivial. However, it preserves the zero mean energies of both qubits, and while the overall entropy increases and mutual information partially fades away, it drives the overall state towards a non-dissipative correlated state with maximally mixed marginals. We proved above that signaling is impossible, even though \mathcal{D}_ρ^A depends not only on a but also on b , and \mathcal{D}_ρ^B on a , which agrees with our no-signaling condition in Eq. (2).

Figure 2 shows a Bloch-ball representation of the time evolutions of the local density operators

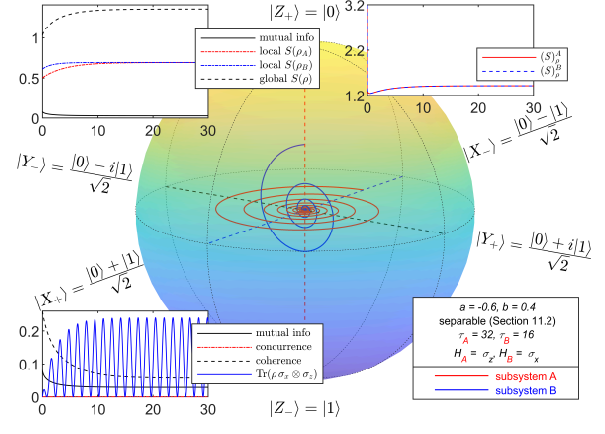


Figure 2: Bloch-ball representation of the time evolutions of the local density operators ρ_A and ρ_B of non-interacting qubits A and B with local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$. The evolution follows the steepest locally perceived entropy ascent equation, Eq. (35), with $\hbar = 1$ and $k_B = 1$. The initial state of the composite system AB is the separable, correlated mixed state given by Eq. (72) with $a = -0.6$ and $b = 0.4$. Insets show the time evolutions of global $[S(\rho)]$, local $[S(\rho_A), S(\rho_B)]$ and locally perceived $[(S_\rho^A) = (S_\rho^B)]$ entropies, mutual information $[S(\rho_A) + S(\rho_B) - S(\rho)]$, concurrence — a coherence measure $[\text{Tr}(\rho^2 H^2 - \rho H \rho H)]$, and the Bell observable $\text{Tr}(\rho \sigma_x \otimes \sigma_z)$. The local evolutions approach the respective local Gibbs states. Without violating the no-signaling condition, dissipation causes a (non complete) reduction in mutual information and coherence, while the Bell observable builds up to a steady state oscillation.

ρ_A and ρ_B for an initial state in the class considered in this section, Eq. (72), with $a = -0.6$ and $b = 0.4$. The time evolution is computed by numerical integration of Eq. (35). The local evolutions of the two non-interacting qubits approach asymptotically the respective local Gibbs states, *i.e.*, the states of maximum local entropy for the given initial mean local energies, for which Eq. (61) reduce to

$$\begin{aligned} \Delta(\text{Bln}(\rho_A)) &= -\beta_A \Delta H_A, \\ \Delta(\text{Bln}(\rho_B)) &= -\beta_B \Delta H_B. \end{aligned} \quad (76)$$

Mutual information, $S(\rho_A) + S(\rho_B) - S(\rho)$, and a measure of coherence, $\text{Tr}(\rho^2 H^2 - \rho H \rho H) = \frac{1}{2} \text{Tr}(K^\dagger K)$ where $K = i[H, \rho]$, are reduced by the SEA dissipation terms, but do not vanish. Indeed, the Bell observable $\text{Tr}(\rho \sigma_x \otimes \sigma_z)$, which is initially zero, builds up to periodic oscillation of constant amplitude.

11.3 Entangled and separable, correlated mixed states

For a slightly more elaborate example that includes entangled mixed states, assume Eq. (63) with $a_x = a_z = a/\sqrt{2}$, $b_x = b_z = b/\sqrt{2}$, and $c_x = c_y = c_z = 2(a - b)/3$, so that the eigenvalues of ρ and those of its partial transpose are

$$\begin{aligned} \lambda_1 &= \frac{1}{4}(1 + a - b), & \lambda_2 &= \frac{1}{12}(3 - a - 5b), \\ \lambda_3 &= \frac{1}{12}(3 + 5a + b), & \lambda_4 &= \frac{1}{12}(3 - 7a + 7b), \end{aligned} \quad (77)$$

and,

$$\begin{aligned} \lambda_1^{PT} &= \frac{1}{12}(3 + a - b), & \lambda_2^{PT} &= \frac{1}{12}(3 - 5a + 5b), \\ \lambda_3^{PT} &= \frac{1}{12}(3 + 2a - 2b + \sqrt{d}), \\ \lambda_4^{PT} &= \frac{1}{12}(3 + 2a - 2b - \sqrt{d}), \end{aligned} \quad (78)$$

with $d = 25a^2 - 14ab + 25b^2$. Figure 3 shows the complete set of admissible pairs of values of parameters a and b for the set of correlated mixed states considered in this example, which encompasses both separable and entangled states. For instance, considering $a = -b$ ($\lambda_2 = \lambda_3$), these states are separable for $-3/14 \leq b \leq 1/4$ and entangled for $1/4 < b \leq 1/2$. We compute explicitly each term in Eq. (57) also for this class of states, to find

$$\begin{aligned} \{(S)^A, \rho_A\} &= -\frac{(1-a^2)(f-5bp)}{20\sqrt{2}}(\sigma_x + \sigma_z), \\ \{(S)^B, \rho_B\} &= \frac{(1-b^2)(g+5ap)}{20\sqrt{2}}(\sigma_x + \sigma_z), \end{aligned} \quad (79)$$

$$\begin{aligned} (H, S)_\rho^A &= -\frac{(1-a^2)(f-5bp)\omega_A}{20\sqrt{2}}(h_x^A + h_z^A), \\ (H, S)_\rho^B &= \frac{(1-b^2)(g+5ap)\omega_B}{20\sqrt{2}}(h_x^B + h_z^B), \end{aligned} \quad (80)$$

$$\begin{aligned} (H, H)_\rho^A &= [1 - \frac{1}{2}(h_x^A + h_z^A)^2 a^2] \omega_A^2, \\ (H, H)_\rho^B &= [1 - \frac{1}{2}(h_x^B + h_z^B)^2 b^2] \omega_B^2, \end{aligned} \quad (81)$$

$$\begin{aligned} -\{\mathcal{D}_\rho^A, \rho_A\} &= -\frac{(1-a^2)(f-5bp)}{5\sqrt{2}[2 - (h_x^A + h_z^A)^2 a^2] k_B \tau_A} \zeta_A, \\ -\{\mathcal{D}_\rho^B, \rho_B\} &= \frac{(1-b^2)(g+5ap)}{5\sqrt{2}[2 - (h_x^B + h_z^B)^2 b^2] k_B \tau_B} \zeta_B. \end{aligned} \quad (82)$$

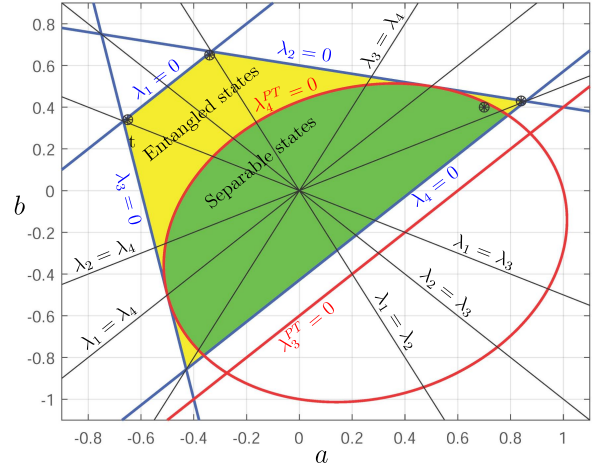


Figure 3: Ranges of admissible values of parameters a and b for the separable and entangled initial states in the class defined in Section 11.3. The four points denoted by the symbol \circledast represent the initial states chosen to illustrate the local time evolutions in Figures 4 and 5. The two points near the $\lambda_1 = 0$ line represent the strongly entangled states for which the panels (b) and (c) of Figure 5 show local entropy decrease for one of the subsystems, while the second-law principle of global entropy non-decrease is not violated.

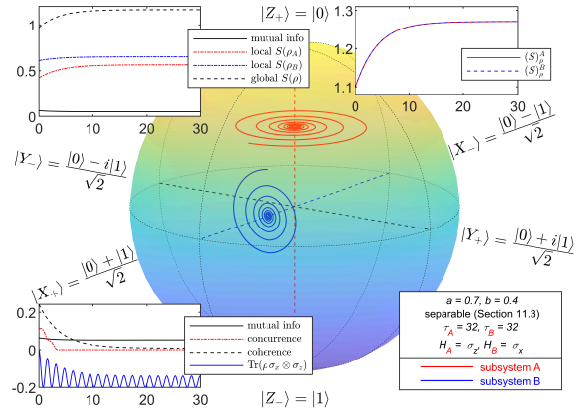


Figure 4: Bloch-ball representation of the time evolutions of the local density operators ρ_A and ρ_B of non-interacting qubits A and B with local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, resulting from numerical integration of the steepest locally perceived entropy ascent equation of motion, Eq. (35). The initial state of the composite system AB is the separable, correlated mixed state in the class of states defined in Section 11.3 with $a = 0.7$ and $b = 0.4$. The local evolutions approach the respective local Gibbs states. Without violating the no-signaling condition, dissipation causes a (non complete) reduction in mutual information and coherence, while the Bell observable converges to a steady state oscillation. Insets as in Figure 2.

Here f , g , and p are defined as

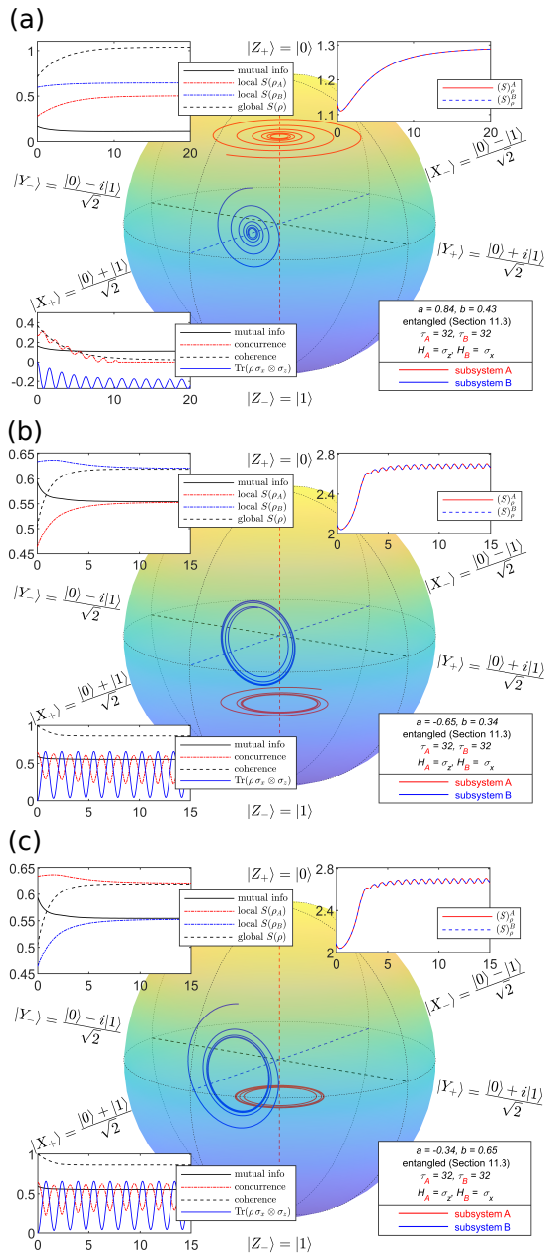


Figure 5: Bloch-ball representations of the time evolution of two non-interacting qubits, obtained from numerical integration of Eq. (35) with initial entangled mixed states from the class defined in Section 11.3. The values of (a, b) in the panels (a), (b), and (c) are $(0.84, 0.43)$, $(-0.65, 0.34)$, and $(-0.34, 0.65)$, respectively. Due to entanglement, local evolutions approach limit cycles with local entropy significantly lower than the corresponding local Gibbs states. In panels (b) and (c), strong entanglement leads to a local entropy decrease in subsystem B and A , respectively, without violating the global entropy non-decrease dictated by the second law. Insets as in Figure 2.

$$\begin{aligned}
 f &= 3\eta_1 - 5\eta_2 + 5\eta_3 - 3\eta_4, \\
 g &= 3\eta_1 + 5\eta_2 - 5\eta_3 - 3\eta_4, \\
 p &= \eta_1 - \eta_2 - \eta_3 + \eta_4,
 \end{aligned} \tag{83}$$

$$\begin{aligned}
 \zeta_A &= \left[1 - \frac{1}{2}(h_x^A + h_z^A)^2 a^2 + \frac{1}{2}(h_x^A + h_z^A) a^2 \right] (\sigma_x + \sigma_z) \\
 &\quad - (h_x^A + h_z^A) \mathbf{h}^A \cdot \boldsymbol{\sigma}_A, \\
 \zeta_B &= \left[1 - \frac{1}{2}(h_x^B + h_z^B)^2 b^2 + \frac{1}{2}(h_x^B + h_z^B) b^2 \right] (\sigma_x + \sigma_z) \\
 &\quad - (h_x^B + h_z^B) \mathbf{h}^B \cdot \boldsymbol{\sigma}_B.
 \end{aligned} \tag{84}$$

For example, if the two non-interacting qubits A and B have local Hamiltonians $H_A = \sigma_z$ and $H_B = \sigma_x$, we find

$$-\{\mathcal{D}_\rho^A, \rho_A\} = -\frac{(1-a^2)(f-5bp)}{5\sqrt{2}(2-a^2)k_B\tau_A} \sigma_x, \tag{85}$$

$$-\{\mathcal{D}_\rho^B, \rho_B\} = \frac{(1-b^2)(g+5ap)}{5\sqrt{2}(2-b^2)k_B\tau_B} \sigma_z, \tag{86}$$

so that again the nonlinear evolution is clearly nontrivial in the sense that the local nonlinear evolution of A (B) does not depend only on ρ_A (ρ_B), despite being no-signaling.

For initial states in the class considered in this section, with parameter values a and b corresponding to the four points in Figure 3, Figures 4 and 5 depict the typical time evolution of the local density operators ρ_A and ρ_B in the local Bloch-balls. These results are obtained from the numerical integration of the steepest locally perceived entropy ascent equation of motion, Eq. (35), for non-interacting subsystems A and B .

Whereas for a separable initial state, Figure 4 shows that the local states approach the respective local Gibbs states, it is clear from Figures 5 that the presence of entanglement does not allow the local states to approach the respective local Gibbs states. Indeed, the local evolutions approach non-dissipative unitary limit cycles where $\{\mathcal{D}^A, \rho_A\} = 0$ and $\{\mathcal{D}^B, \rho_B\} = 0$, and therefore Eqs. (61) are satisfied. Without violating the no-signaling condition, dissipation causes a (non complete) reduction in mutual information and coherence, while the Bell observable $\text{Tr}(\rho \sigma_x \otimes \sigma_z)$ and concurrence converge to steady state oscillations.

12 Conclusion

In this work, we explored the no-signaling principle and its implications for nonlinear dynamics

in quantum theory. We examined the measure-theoretic representation of mixed ensembles, by providing a consistent framework to handle nonlinearities and describe ensemble mixing unambiguously. This approach also offers a conceptual basis for developing nonlinear quantum dynamical models for non-equilibrium systems and addressing unresolved questions about individual quantum states. Additionally, we reviewed the original philosophical motivations underlying the steepest entropy ascent (SEA) nonlinear evolution law, which trades linearity for strong thermodynamic consistency.

Our analysis demonstrates that the SEA formalism provides a valid framework for describing evolution or designing master equations. These can be applied in nonlinear extensions of quantum mechanics (QM) or in phenomenological models of open quantum systems within quantum thermodynamics. Our key focus was the no-signaling principle. We introduced a criterion for no-signaling based on local perception operators (LPOs), a generalization of the traditional density-operator-based definition. Unlike the conventional assumption that local evolution depends solely on local properties, the SEA formalism incorporates the local effects of preexisting correlations and coherence through LPOs. Our detailed examination of the foundations and properties of LPOs highlights their potential as key tools for constructing signaling-free, nonlinear, and non-local non-equilibrium dynamical models, even outside of the SEA framework.

By inherently respecting the no-signaling condition, the SEA formalism upholds the principle of causality and aligns with the axiomatic paradigm proposed by Popescu and Rohrlich [86] for quantum theories. Furthermore, the SEA approach addresses conceptual challenges such as the Schrödinger-Park paradox, offering insights into a thermodynamically consistent integration of quantum mechanics and non-equilibrium thermodynamics. Related to the no-signaling property, our discussion of LPOs and supporting numerical examples strengthen the conjecture — supported by prior numerical results from Cano-Andrade *et al.* [26] — that SEA evolution is universally compatible with decoherence principles. Specifically, SEA evolution ensures that correlations and entanglement cannot build up between non-interacting systems but can decay partially

or completely, a process mediated by LPOs.

An intriguing implication of the SEA formalism is its compatibility with thermodynamic principles while allowing for scenarios where, under strong entanglement, the local entropy of a subsystem may temporarily decrease during the evolution, without violating the no-signaling condition nor the global entropy non-decrease dictated by the second law.

We hope this study provides a foundational basis for exploring applications of SEA modeling in quantum computing. Our comprehensive approach and observations aim to stimulate constructive discussions and inspire further studies, particularly in developing thermodynamically consistent models for non-equilibrium processes. These findings have potential applications in modeling the dynamics of quantum systems, advancing quantum technologies, and contributing to the framework of quantum thermodynamic resource theories, even beyond the SEA paradigm.

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